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POSITIVELY CURVED COMPLEX SUBMANIFOLDS IMMERSED IN A COMPLEX PROJECTIVE SPACE

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1. Statement of results

Let $P_{n+p}(C)$ be a complex projective space of complex dimension $n+p$ with the Fubini-Study metric of constant holomorphic sectional curvature 1. By a *Kaehler submanifold* we mean a complex submanifold with induced Kaehler structure.

The purpose of this paper is to prove the following two theorems.

Theorem 1. *Let M be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If every holomorphic sectional curvature of M is greater than $1/2$, and the scalar curvature of M is constant, then M is totally geodesic in $P_{n+p}(C)$.*

Theorem 2. *Let M be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If every holomorphic sectional curvature of M is greater than $1 - \frac{1}{2}(n+2)/(n+2p)$, then M is totally geodesic in $P_{n+p}(C)$.*

It is clear that in the case of $p = 1$, Theorem 2 is an improvement of Theorem 1.

2. Preliminaries

Let J (resp. $\tilde{J}$) be the complex structure of M (resp. $P_{n+p}(C)$), let g (resp. $\tilde{g}$) be the Kaehler metric of M (resp. $P_{n+p}(C)$), and denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y ,$$

and satisfies $\tilde{J}\sigma(X, Y) = \sigma(JX, Y) = \sigma(X, JY)$, and the structure equation of Gauss is

$$\begin{aligned} g(R(X, Y)Z, W) = {} & \tilde{g}(\sigma(X, W), \sigma(Y, Z)) - \tilde{g}(\sigma(X, Z), \sigma(Y, W)) \\ & + \tfrac{1}{4}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \end{aligned}$$

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$$+ g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) + 2g(X, JY)g(JZ, W)] ,$$

where R is the curvature tensor field of M. Let $\xi_1, \cdots, \xi_p, \xi_{1*}, \cdots, \xi_{p*}$ ($\xi_{i*} = J\xi_i$) be local fields of orthonormal vectors normal to M. We use the following convention on the range of indices: $i, j = 1, \cdots, p$; $\lambda, \mu = 1, \cdots, p, 1^*, \cdots, p^*$. If we set

$$g(A_\lambda X, Y) = \tilde{g}(\sigma(X, Y), \xi_\lambda) ,$$

then A_λ, $\lambda = 1, \cdots, p, 1^*, \cdots, p^*$, are local fields of symmetric linear transformations. We can easily see that $A_{i*} = JA_i$ and $JA_i = -A_iJ$ so that, in particular, $\operatorname{tr} A_\lambda = 0$. Moreover, the structure equation of Gauss can be written in terms of A_λ's as

$$\begin{aligned} g(R(X, Y)Z, W) = {} & \sum [g(A_\lambda X, W)g(A_\lambda Y, Z) - g(A_\lambda X, Z)g(A_\lambda Y, W)] \\ & + \tfrac{1}{4}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & \quad + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ & \quad + 2g(X, JY)g(JZ, W)] . \end{aligned} \tag{1}$$

Let S be the Ricci tensor of M, and ρ the scalar curvature of M. Then we have

$$S(X, Y) = \tfrac{1}{2}(n + 1)g(X, Y) - 2g(\textstyle\sum A_i^2 X, Y) , \tag{2}$$

$$\rho = n(n + 1) - \|\sigma\|^2 , \tag{3}$$

where $\|\sigma\|$ is the length of the second fundamental form of the immersion so that

$$\|\sigma\|^2 = 2 \sum \operatorname{tr} A_i^2 .$$

We can see from (1) that the holomorphic sectional curvature H of M determined by a unit vector X is given by

$$H(X) = 1 - 2\|\sigma(X, X)\|^2 = 1 - 2 \sum g(A_\lambda X, X)^2 . \tag{4}$$

It is known that the second fundamental form σ satisfies a differential equation which gives

Lemma 1 [2]. *We have*

$$\begin{aligned} \tfrac{1}{2}\Delta\|\sigma\|^2 = {} & \|\nabla'\sigma\| + \sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 \\ & - \sum [\operatorname{tr}(A_\lambda A_\mu)]^2 + \tfrac{1}{2}(n + 2)\|\sigma\|^2 , \end{aligned}$$

where Δ denotes the Laplacian, and ∇' the covariant differentiation with respect to the connection (in tangent bundle) $\oplus$ (normal bundle).

3. Proof of theorems

Since M is complete and every holomorphic sectional curvature of M is bounded from below by a positive number, M is compact.

First we prove Theorem 1. Since $1/2 < H \leq 1$ and ρ is constant, Theorem 2 in [1] implies that H is constant. This, combined with the corollary to Theorem 3 in [4] and Theorem 1 in [3], implies that M is totally geodesic.

Next we prove Theorem 2. From (4) we can see that if every holomorphic sectional curvature of M is greater than $1 - \delta$, then the square of every eigenvalue of A_λ must be smaller than $\delta/2$. Therefore we have

$$\text{(5)} \qquad \operatorname{tr}(A_\lambda^2 A_\mu^2) \leq \frac{\delta}{2} \operatorname{tr} A_\lambda^2 \qquad \textit{for all } \lambda \textit{ and } \mu .$$

Lemma 2. *If $H > 1 - \delta$, then*

$$\text{(6)} \qquad \sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 + 2p\delta \|\sigma\|^2 \geq 0 .$$

Proof. We have

$$\begin{aligned}
&\sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 \\
&\quad = -2 \sum \operatorname{tr}(A_\lambda^2 A_\mu^2 - (A_\lambda A_\mu)^2) \\
&\quad = -2\Big[\sum_{i \neq j} \operatorname{tr}(A_i^2 A_j^2 - (A_i A_j)^2) + 2 \sum \operatorname{tr}(A_i^2 A_{i*}^2 - (A_i A_{i*})^2) \\
&\qquad\qquad + \sum_{i \neq j} \operatorname{tr}(A_i^2 A_{j*}^2 - (A_i A_{j*})^2) + \sum_{i \neq j} \operatorname{tr}(A_{i*}^2 A_{j*}^2 - (A_{i*} A_{j*})^2)\Big] \\
&\quad = -4\Big[\sum_{i \neq j} \operatorname{tr}(A_i^2 A_j^2 - (A_i A_j)^2) + 2 \sum \operatorname{tr} A_i^4 + \sum_{i \neq j} \operatorname{tr}(A_i^2 A_j^2 + (A_i A_j)^2)\Big] \\
&\quad = -8\Big[\sum_{i \neq j} \operatorname{tr} A_i^2 A_j^2 + \sum \operatorname{tr} A_i^4\Big] = -8 \sum \operatorname{tr}(A_i^2 A_j^2) .
\end{aligned}$$

From (5) it follows that

$$\sum \operatorname{tr}(A_i^2 A_j^2) \leq \frac{p\delta}{2} \sum \operatorname{tr} A_i^2 = \frac{p\delta}{4} \|\sigma\|^2 .$$

which implies (6) immediately.

Lemma 3. *If $H > 1 - \delta$, then*

$$\text{(7)} \qquad \sum [\operatorname{tr}(A_\lambda A_\mu)]^2 \leq n\delta \|\sigma\|^2 .$$

Proof. Let $\Lambda = \operatorname{tr}(A_\lambda A_\mu)$. Then Λ is a local field of symmetric $(2p, 2p)$-matrix. Since $\sum [\operatorname{tr}(A_\lambda A_\mu)]^2 = \operatorname{tr} \Lambda^2$, $\sum [\operatorname{tr}(A_\lambda A_\mu)]^2$ is a geometric invariant, i.e., it does not depend on the choice of $\xi_1, \cdots, \xi_p$. Therefore it suffices to show that the inequality holds for a suitable choice of $\xi_1, \cdots, \xi_p$ at each point of M. Since $\Lambda = \mathrm{tr}(A_\lambda A_\mu)$ is a real representation of the Hermitian matrix $\Lambda_0 = (\mathrm{tr}(A_i A_j) + \sqrt{-1}\,\mathrm{tr}(A_i A_{j*}))$, it can be diagonalized by a unitary transformation at each point of M. In other words, at each point of M, Λ can be assumed to be diagonal for a suitable choice of $\xi_1, \cdots, \xi_p$, that is,

$$
{}^tU\Lambda U = \begin{bmatrix} \mathrm{tr}\,\tilde{A}_1^2 & & & & & \\ & \ddots & & & 0 & \\ & & \mathrm{tr}\,\tilde{A}_p^2 & & & \\ & & & \mathrm{tr}\,\tilde{A}_1^2 & & \\ & 0 & & & \ddots & \\ & & & & & \mathrm{tr}\,\tilde{A}_p^2 \end{bmatrix}
$$

for (real representation of) some unitary matrix U. Therefore we obtain

$$
(8) \qquad \sum [\mathrm{tr}(A_\lambda A_\mu)]^2 = \mathrm{tr}\,\Lambda^2 = \mathrm{tr}({}^tU\Lambda U)^2 = 2\sum(\mathrm{tr}\,\tilde{A}_i^2)^2 \leq 4n \sum \mathrm{tr}\,\tilde{A}_i^4\,,
$$

by using the general fact that a symmetric $(2n, 2n)$-matrix A satisfies $(\mathrm{tr}\,A^2)^2 \leq 2n\,\mathrm{tr}\,A^4$. (8), together with (5), hence implies (7). q.e.d.

From Lemmas 1, 2 and 3 it follows that

$$
\tfrac{1}{2}\Delta\|\sigma\|^2 \geq [\tfrac{1}{2}(n+2) - (n+2p)\delta]\,\|\delta\|^2\,.
$$

Since $\delta = \frac{1}{2}(n+2)/(n+2p)$, we have $\Delta\|\sigma\|^2 \geq 0$. Thus by the well-known Bochner's lemma, $\|\delta\|^2$ is constant, and so is ρ due to (3). Since $1 - \frac{1}{2}(n+2)/(n+2p) \geq \frac{1}{2}$, Theorem 1 implies that M is totally geodesic.

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